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SHORTER COMMUNICATIONS

A NEW APPROACH TO THE CONCEPT OF EFFICIENCY OF HIGH FINS

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NOMENCLATURE

- e, thickness of the fin [m];
- h, height of the fin [m];
- k, heat conductivity of the fin [W/m K];
- L, width of the fin [m];
- Pc, heat capacity rate [W/K];
- $\dot{Q}$, heat flux received by the fluid with lower Pc, or in a transverse section of the fin [W];
- S, current heat-transfer surface [m^2];
- S^*, total exchange surface [m^2];
- T, local temperature of the fluid with higher heat capacity rate, or of the fin [K];
- t, local temperature of the fluid with lower heat capacity rate [K];
- U, overall heat-transfer coefficient of the exchanger, or heat-transfer coefficient at the fin [W/m^2 K];
- Bi, Biot number $\left(=\dfrac{Ue}{k}\right)$;
- H, height of the fin ($= h/e$);
- p_1, p_2, roots of a characteristic equation;
- R, roots difference of the characteristic equation.

Greek symbols

- β, coefficient defined by equation (5);
- ε, temperature of the fluid with lower heat capacity rate $\left(=\dfrac{t-t_1}{T_1-t_1}\right)$;
- ε^*, effectiveness of the heat-transfer cell $\left(=\dfrac{t_2-t_1}{T_1-t_1}\right)$;
- ϑ, temperature of the fluid with higher heat capacity rate, or of the fin $\left(=\dfrac{T-t_1}{T_1-t_1}\right)$;
- ρ, heat capacity rates ratio $\left(=\dfrac{(Pc)_{\min}}{(Pc)_{\max}}\right)$;
- σ, current heat-transfer active surface ($= US/(Pc)_{\min}$);
- σ^*, total active surface ($= US^*/(Pc)_{\min}$).

Subscripts

- 1, denote inlet value;
- 2, denote outlet value;
- 0, denote the section of the fin at $\sigma = 0$;
- t, denote the tip of the fin;
- min, lower value;
- max, higher value.

IT IS well known that the method of "effectiveness-number of transfer units" often appears to be the most advantageous in order to evaluate the performances of a two-fluids heat exchanger. We shall prove in this short note that the same concept and method can apply to evaluate the heat-transfer rate and to calculate temperature profiles of arrays of high fins with finite flow of fluid between them.

THE PRELIMINARY CASE OF A PARALLEL-FLOW HEAT EXCHANGER

Dimensionless variables ε and σ are defined in such a way that their values ε^* and σ^* at the end of the heat exchanger are in fact the effectiveness and the number of transfer units.

At first, we shall consider briefly the simple case of parallel flow heat exchanger (Fig. 1), in order to show the successive steps of the method. We may write two local equations. The first one is the transfer equation:

$$\frac{d\dot{Q}}{dS} = U(T-t),$$

or in dimensionless form:

$$\frac{d\varepsilon}{d\sigma} = \vartheta - \varepsilon. \tag{1}$$

The second equation is a local energy balance:

$$\rho\, dt = -dT$$

which may be integrated and written as:

$$\vartheta = 1 - \rho\varepsilon. \tag{2}$$

By elimination of ϑ between (1) and (2) and after integration, we get:

$$\varepsilon = \frac{1 - e^{-(1+\rho)\sigma}}{1+\rho}. \tag{3}$$

This equation describes the temperature profile of one of the fluids vs the dimensionless active surface (local number of transfer units), with the heat capacity rates ratio as a parameter. When we put stars upon ε and σ, equation (3) gives then the so-called effectiveness of the heat exchanger.

The ϑ profile may be obtained from (2) and (3), to give:

$$\vartheta = \frac{1 + \rho\, e^{-(1+\rho)\sigma}}{1+\rho}.$$

The common limit value for ε and ϑ when $\sigma \to \infty$ appears immediately to be $(1+\rho)^{-1}$.

FINS WITH FINITE FLUID FLOW BETWEEN THEM

Frequently the fins (either longitudinal or transverse high fins) belonging to a heat exchanger may be idealized as follows: the fins are rectangular; each fin base is at a constant temperature (this of the tube on which it is fastened); the streamlines in the fluid and the fluxlines of heat in the core of the fin are parallel, but may have either the same or opposite directions.

This problem can be treated in a very analogous manner as the preceding one. But here, the two transfer media act in very different ways: the one (the fluid) possesses pure convective properties in x or σ direction; the other (the metallic core of the fin) acts only by conduction.

Now ε, ϑ and σ are defined in the same way as before, except that there exists only one heat capacity rate Pc. Therefore ρ does not exist anymore. U becomes a simple

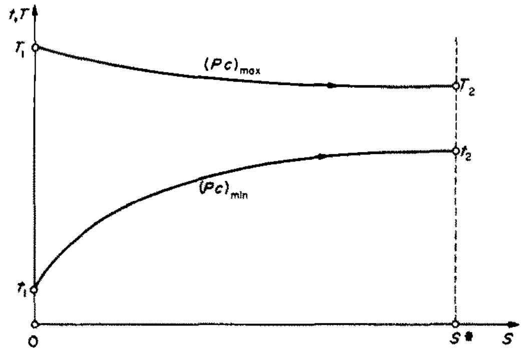

FIG. 1. Schematic diagram of evaluation of temperatures in a co-current heat exchanger.

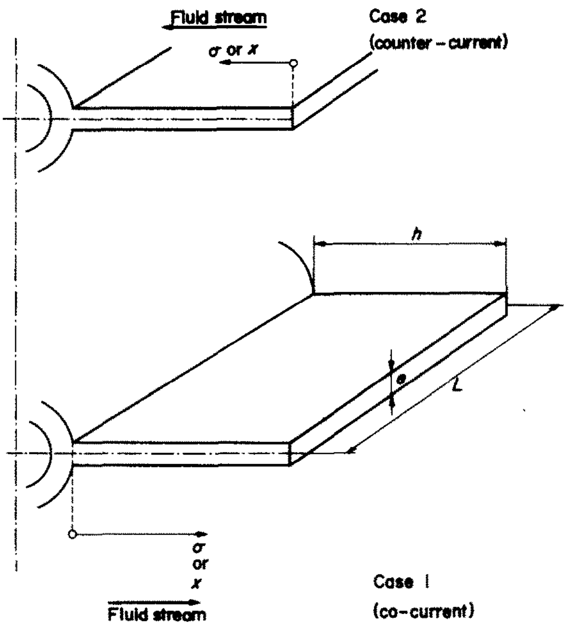

FIG. 2. Geometrical configuration for the fin problem.

convection coefficient; the area of contact between fin and fluid is $S = 2Lx$ (Fig. 2).

The local transfer equation is now again equation (1), but the local energy balance writes as:

$$\beta \frac{d^2\vartheta}{d\sigma^2} = \frac{d\varepsilon}{d\sigma}, \tag{4}$$

whatever may be the respective directions of heat flux and fluid flow.

Dimensionless coefficient β is equal to:

$$\beta = \frac{2kUeL^2}{(Pc)^2} = \frac{\sigma^{*2}}{2BiH^2}. \tag{5}$$

σ^* has the form of a Stanton number; H is the dimensionless height of the fin.

After a first integration of (4) and using the condition $\varepsilon = 0$ at boundary $\sigma = 0$, we get:

$$\varepsilon = \beta\left[\frac{d\vartheta}{d\sigma} - \left(\frac{d\vartheta}{d\sigma}\right)_{\sigma=0}\right]. \tag{6}$$

If we define the dimensionless heat flux in the transverse section of the fin at $\sigma = 0$, as:

$$\dot{\varphi}_0 = \frac{\dot{Q}_0}{Pc(T_1 - t_1)},$$

Fourier's law may be written in the form

$$\dot{\varphi}_0 = -\beta\left(\frac{d\vartheta}{d\sigma}\right)_{\sigma=0}. \tag{7}$$

By combination of equations (1), (6) and (7), we have:

$$\frac{d^2\varepsilon}{d\sigma^2} + \frac{d\varepsilon}{d\sigma} - \frac{\varepsilon}{\beta} = -\frac{\dot{\varphi}_0}{\beta}. \tag{8}$$

Integration of this differential equation is obvious. The roots of the characteristic equation are:

$$p_1 = -\tfrac{1}{2}(1+R), \qquad p_2 = -\tfrac{1}{2}(1-R),$$

with

$$R = \sqrt{\left(1 + \frac{4}{\beta}\right)}.$$

We must now distinguish two sets of boundary conditions.

1. *Streamlines and fluxlines are in the same direction*

B.C.: $\qquad \varepsilon = 0, \quad \sigma = 0,$

$$\vartheta = \varepsilon + \frac{d\varepsilon}{d\sigma} = 1, \qquad \sigma = 0.$$

We have here:

$$\varepsilon = \frac{(1+p_2\dot{\varphi}_0)e^{p_1\sigma}}{p_1 - p_2} - \frac{(1+p_1\dot{\varphi}_0)e^{p_2\sigma}}{p_1 - p_2} + \dot{\varphi}_0. \tag{9}$$

At $\sigma = 0$, $\dot{\varphi}_0$ is the total heat flux crossing the base of the fin, i.e. comprising the heat flux transmitted to the surrounding fluid and the flux crossing through the tip of the fin ($\dot{\varphi}_t$):

$$\dot{\varphi}_0 = \varepsilon^* + \dot{\varphi}_t.$$

From (9), in terms of σ^* and β (or R), we have for ε^*:

$$\varepsilon^* = \frac{2}{1 + \dfrac{R}{\tanh\dfrac{R\sigma^*}{2}}} - \dot{\varphi}_t\left(1 + \frac{Re^{\sigma^*/2}}{\sinh\dfrac{R\sigma^*}{2} + R\cosh\dfrac{R\sigma^*}{2}}\right). \tag{10}$$

2. *Streamlines and fluxlines are in opposite directions*

B.C.: $\qquad \varepsilon = 0, \quad \sigma = 0,$

$$\vartheta = \varepsilon + \frac{d\varepsilon}{d\sigma} = 1, \qquad \sigma = \sigma^*.$$

Here, $\dot{\varphi}_0$ reduces to $\dot{\varphi}_t$, and we get by integration of (8):

$$\varepsilon = \frac{[1 - \dot{\varphi}_t(1 + p_1 e^{p_2\sigma^*})]e^{p_1\sigma} - [1 - \dot{\varphi}_t(1 + p_2 e^{p_1\sigma^*})]e^{p_2\sigma}}{p_1 e^{p_2\sigma^*} - p_2 e^{p_1\sigma^*}} + \dot{\varphi}_t. \tag{11}$$

We can deduce the expression for ε^*:

$$\varepsilon^* = \frac{2}{1 + \dfrac{R}{\tanh\dfrac{R\sigma^*}{2}}} + \dot{\varphi}_t\left(1 - \frac{R\,e^{-\sigma^*/2} + 2\sinh R\sigma^*/2}{\sinh\dfrac{R\sigma^*}{2} + R\cosh\dfrac{R\sigma^*}{2}}\right). \tag{12}$$

Comparison of equations (10) and (12) shows immediately that when the heat flux through the tip of the fin is zero, the expressions for ε^* are the same; the heat fluxes transferred to the surrounding fluid are thus equal.

When $\dot{\varphi}_t$ is not zero, analogous losses through the tip of the fins for the two cases considered here result in values of $\dot{\varphi}_t$ to be put into (10) and (12) which are equal but with opposite signs. If this is done, one can easily verify that ε^* for the "co-current" case is always lower than for the "counter-current" case.

The temperature profiles in the fin can be calculated easily from equation (1) and either equation (9) or (11).

THE EFFICIENCY OF THE FIN

In what follows, we shall restrict ourselves to the case where there is no heat loss at the tip of the fin; the direction of fluid flow needs not to be indicated anymore.

The classical concept of efficiency introduced by Harper and Brown [1] can now be introduced: this "constitution efficiency" ε_k is the ratio of the heat flux dissipated by the fin to the one which would be transferred if the heat conductivity of the fin had become infinite. If $k \to \infty$, then $Bi \to 0$, $R \to 1$, and we have:

$$\varepsilon_k = \frac{1+\dfrac{1}{\tanh\dfrac{\sigma^*}{2}}}{1+\dfrac{R}{\tanh\dfrac{R\sigma^*}{2}}}.$$

When, furthermore, the heat capacity rate becomes infinite ($\sigma^* \to 0$), so as it is supposed in most cases, we find the classical expression:

$$\lim_{Pc\to\infty} \varepsilon_k = \frac{\tanh H\sqrt{(2Bi)}}{H\sqrt{(2Bi)}}.$$

But we can find other relevant criteria of perfectness, which may characterize the property of efficiency of a fin. Particularly, the parameter ε^* that we have emphasized here above appears to be a "global efficiency" of the fin, since the fin of reference should have both infinite thermal conductivity and height or extension.

Another interesting reference should be a fin of infinite extension but with its real thermal conductivity: the "extension efficiency" defined in this scope is of interest for the designer [2].

With finite fluid flow, the "extension efficiency" ε_h is given by:

$$\varepsilon_h = \frac{1+R}{1+\dfrac{R}{\tanh\dfrac{R\sigma^*}{2}}},$$

while, when fluid flow becomes infinite, the following and very simple expression is found:

$$\lim_{Pc\to\infty} \varepsilon_h = \tanh H\sqrt{(2Bi)}.$$

CONCLUSIONS

We have solved the most general problem of heat transfer from fins with heat losses at their tips and finite fluid flow between them. This was done in the scope of the concept of effectiveness of a heat-transfer cell. From the results first obtained, we have deduced general expressions for several parameters of efficiency of a fin.

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ON THE LINEARIZED ANALYSIS OF ENTRANCE FLOW IN HEATED, POROUS CONDUITS

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NOMENCLATURE

C_n, D_n, constant coefficients;
$E_1(x), E_2(x)$, residual terms;
$F_n(y)$, function defined by (13);
Nu, Nusselt number;
Pe, Peclet number;
Pr, Prandtl number;
Re, Reynolds number;
T, temperature;
U, mean velocity;
u, v, velocity components in x, y directions;
x, y, coordinates parallel to and normal to flow direction;
ξ, reduced coordinate of x, (9);
λ_n, β_n, eigenvalues, (18), (27);
ϕ, confluent hypergeometric function.

Superscripts
b, bulk;
i, inlet of conduit;
w, wall of conduit.

THE STEADY, laminar, incompressible flow in the entrance region of heated, porous conduits is investigated by the linearized method, which is known to yield good results for momentum transfer in tubes of impermeable wall but is only fairly good for momentum transfer and failed for heat transfer in porous tubes [1]. It is shown in this note that the method gives also good results for porous conduits and analytical solutions can be obtained, provided that the transverse velocity is taken into account and suitably approximated.

LINEARIZED EQUATIONS

Consider the laminar flow of an incompressible fluid between two parallel semi-infinite porous plates and through a semi-infinite circular tube of porous wall. As usual, all thermo-physical properties of the fluid are assumed constant, and the rate of injection or suction and the wall temperatures are assumed constant and uniform. The inlet velocity and temperature profiles are prescribed: uniformly distributed over the cross-section or fully-developed in conduits